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Catalan Tau Collocation for Numerical Solution of 2-Dimentional Nonlinear Partial Differential Equations

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ABSTRACT
Tau method which is an economized polynomial technique for solving ordinary and partial differential equations with smooth solutions is modified in this paper for easy computation, accuracy and speed. The modification is based on the systematic use of 'Catalan polynomial' in collocation tau method and the linearizing the nonlinear part by the use of Adomian's polynomial to approximate the solution of 2-dimentional Nonlinear Partial differential equation. The method involves the direct use of Catalan Polynomial in the solution of linearizedPartial differential Equation without first rewriting them in terms of other known functions as commonly practiced. The linearization process was done through adopting the Adomian Polynomial technique. The results obtained are quite comparable with the standard collocation tau methods for nonlinear partial differential equations.
***KEYWORDS**:* Tau method, Collocation tau method, partial differential equation, Catalan Polynomial. Nonlinear

I. INTRODUCTION

In his memoir of 1938, Lanczos introduced the use of Chebyshev polynomials in relation to the solution of linear differential equation with polynomial coefficients in terms of finite expansions of the form.

$$Dy(x) = 0 \qquad (1)$$

Since then scholars have developed this method in different ways and have found a wide field of applications, because they are specially designed to provide economized representations for considerable number of functions frequently used in scientific computation (Liu *et al*. 2003).These are derivable from linear differential equations with polynomial coefficients. Ortiz and Samara (1984) worked on the solution of PDE's with variable coefficient using an operational approach and the result was encouraging .Odekunle(2006) also used Catalan polynomial basis to find solutions to ordinary differential equation and the result converges faster than the existing methods. Odekunle et al. (2014) also used Catalan polynomial basis to formulate solution to 2-dimentional linear PDE's.

In their own work Sam and Liu (2004) extended the tau collocation method for solving ordinary differential equation to the solution of partial differential equations defined on a finite domain with initial, boundary and mixed condition using Chebyshev polynomial as basis function and they arrived at a beautiful result

In this work, we shall follow the approach of Sam and Liu (2004) and Odekunle (2006) to determine the approximate solution of a 2-dimentional nonlinear partial differential equation on a finite domain using Catalan polynomial as the perturbation term. We shall also use multiple choice of perturbation term to overcome the problem of over-determination in the resulting system of equations encountered in collocation tau method. The conversion of the partial differential equations to system of equation was effectively done using Kronecker product.

II. TAU-COLLOCATION METHOD FOR2-DIMENSIONAL LINEAR PDEs

Definition 1: Catalan Polynomial (Odekunle, 2006)

We define Catalan polynomial $C_*(x)$ as

$$C_{\bullet}(x) = \sum_{i=0}^{n} [\frac{1}{1+i}\binom{2i}{i}]x^i, \ i = 0,1,2,\ldots$$

Where

$$\binom{i}{k} = \frac{i!}{k!(i-k)!}, \ i,k = 0,1,2,\ldots$$

Definition 2: Kronecker Product (Graham, 1981).

The kronecker product, denoted by $\otimes$,is an operation on two matrices of arbitrary size resulting in a block matrix. It gives the matrix of the tensor product with respect to a standard choice of basis.
In the development of the method, let

$$u_{n_x n_y} = \sum_{i_x=0}^{n_x} \sum_{i_y=0}^{n_y} a_{i_x i_y} x^{i_x} y^{i_y} \quad (2)$$

be a polynomial with n_x^{th} degree in x *and* n_y^{th} degree in y is substituted into a given PDE below

$$Lu(x, y) = f(x, y),\ (x, y) \in [a_x, b_x] \times [a_y, b_y] \qquad (3)$$

Subject to the supplementary conditions

$$D_{y_p}\Big|_{x=x_p} u(x, y) = \sigma_{y_p}(y),\ p = 1(1)N_x \qquad (4)$$

$$D_{x_q}\Big|_{y=y_q} u(x, y) = \sigma_{x_q}(y),\ q = 1(1)N_y \qquad (5)$$

Where N_x and N_y are positive constants, L class of linear PDE's in two variables x *and* y and D_{x_q}, D_{y_p} are linear partial differential operators in x *and* y respectively.
When (2) is substituted into problem (3) to (5), an over-determined system of linear algebraic equations with $(n_x+1)(n_y+1)$ unknown coefficient $a_{i_x i_y}$ *where* $i_x = 0(1)n_x$ *and* $i_y = 0(1)n_y$ are formed.

Let $\tau_x = (\tau_{x0}, \tau_{x1}, ..., \tau_{x\varphi x-1})$ *and* $\tau_y = (\tau_{y0}, \tau_{y1}, ..., \tau_{y\varphi y-1})$ be the φ_x *and* φ_y free parameters respectively. A perturbation term $H_{n_x n_y}(x, y)$ with unknown parameters τ_x *and* τ_y is added to the right-hand side of equation (3) so as to construct a balanced system of linear algebraic equations for determining the approximate polynomial solution $u_{n_x n_y}(x, y)$. Then equation (3) and conditions (4) and (5) becomes

$$Lu_{n_x n_y}(x, y) = f(x, y) + H_{n_x n_y}(x, y),\ (x, y) \in [a_x, b_x] \times [a_y, b_y] \qquad (6)$$

Subject to the supplementary conditions

$$D_{y_p}\Big|_{x=x_p} u_{n_x n_y}(x, y) = \sigma_{y_p}(y), \quad p = 1(1)N_x \qquad (7)$$

$$D_{x_q}\Big|_{y=y_q} u_{n_x n_y}(x, y) = \sigma_{x_q}(y), \quad q = 1(1)N_y \qquad (8)$$

This is defined as the associated tau problem to equations (3) to (5). As with tau-Collocation for ODEs, the format of perturbation term in equation (6) is chosen as

$$H_{n_x n_y}(x, y) = g_{n_x n_y}(x, y; \tau_x, \tau_y) V_{x, n_x - N_x + 1}^{[a_x, b_x]}(x) V_{y, n_y - N_y + 1}^{[a_y, b_y]}(y) \quad (9)$$

Where $V_{x, n_x - N_x + 1}^{[a_x, b_x]}(x)$ and $V_{y, n_y - N_y + 1}^{[a_y, b_y]}(y)$ are Catalan polynomials of degree $(n_x - N_x + 1)$ defined on $[a_x, b_x]$ and $(n_y - N_y + 1)$ defined on $[a_y, b_y]$ respectively.
The formulation of the Tau-collocation method for 2-dimensional PDEs is divided conceptually into two parts. They are: (i) the formulation of the linear PDE and (ii) the formulation of the conditions of the given problem.
III. FORMULATION OF THE LINEAR PDE FOR THETAU PROBLEM
Following Sam and Liu (2004),

$$\Pi_{n_x n_y}(x, y) = \sum_{r_x=0}^{v_x} \sum_{r_y=0}^{v_y} (\underline{y_{n_y}}(\eta_{n_y}^{r_y})' \otimes q_{r_x r_y}(x, y) \underline{x_{n_x}}(\eta_{n_x}^{r_x})') vec(A_{n_x n_y}) \qquad (10)$$

Substituting (10) into equation (6) gives

$$\Pi_{n_x n_y}(x, y) vec(A_{n_x n_y}) = f(x, y) + H_{n_x n_y}(x, y) \quad (11)$$

Where

$$H_{n_x n_y}(x, y) = g_{n_x n_y}(x, y; \tau_x, \tau_y) V_{x, n_x - N_x + 1}^{[a_x, b_x]}(x) V_{y, n_y - N_y + 1}^{[a_y, b_y]}(y),$$

Is the perturbation term of the Catalan polynomial with $(n_x - N_x + 1)(n_y - N_y + 1)$ zeros. Collocating (11) at $(x_{i_x}, y_{i_y}), i_x = 0(1)n_x - N_x \; and \; i_y = 0(1)n_y - N_y$, gives

$$\Gamma Vec(A_{n_x n_y}) = \underline{F} \quad (12)$$

where

$$\Gamma = \begin{pmatrix} \Pi_{n_x n_y}(x_0, y_0) \\ \Pi_{n_x n_y}(x_1, y_1) \\ . \\ . \\ . \\ \Pi_{n_x n_y}(x_0, y_{n_y - N_y}) \\ \Pi_{n_x n_y}(x_1, y_0) \\ . \\ . \\ . \\ \Pi_{n_x n_y}(x_{n_x - N_x}, y_{n_y - N_y}) \end{pmatrix} \qquad \underline{F} = \begin{bmatrix} f(x_0, y_0) \\ f(x_0, y_1) \\ . \\ . \\ . \\ f(x_0, y_{n_y - N_y}) \\ . \\ . \\ . \\ f(x_{n_x - N_x}) \end{bmatrix}$$

Equation (6) is now successfully converted to a set of linear algebraic equations.

IV. FORMULATION OF THE CONDITIONS OF THE TAU PROBLEM

The following steps show the formulation of N_x conditions (7) of the Tau problem (6) to (8), ie

$$\Pi_{y_p n_x n_y} = \sum_{r_x=0}^{k_x} \sum_{r_y=0}^{k_y} (\underline{\underline{y_{n_y}}}(\eta_{n_y}^{r_y})' \otimes \theta_{r_x r_y}(x, y) \underline{\underline{x_{n_x}}}(\eta_{n_x}^{r_x})') vec(A_{n_x n_y}) \quad (13)$$

Substituting (13) into equation (7) gives

$$\Pi_{y_p n_x n_y}(x_p, y) vec(A_{n_x n_y}) = \sigma_{y_p}(y), \;\; p = 1(1)N_x \quad (14)$$

Let $y_{i_y}, i_y = 0(1)n_y$ be the $n_y + 1$ zeros of the polynomial $V_{y, n_y + 1}^{[a_y, b_y]}$. By collocating these $n_y + 1$ zeros into equation (14), we have

$$\Gamma_y vec(A_{n_x n_y}) = \underline{F_y} \quad (15)$$

where

$$\Gamma_y = \begin{pmatrix} \Pi_{y_1 n_x n_y}(x_1, y_0) \\ \Pi_{y_1 n_x n_y}(x_1, y_1) \\ . \\ . \\ . \\ \Pi_{y_1 n_x n_y}(x_1, y_n) \\ \Pi_{y_2 n_x n_y}(x_2, y_0) \\ . \\ . \\ . \\ \Pi_{y_{N_x} n_x n_y}(x_{N_x}, y_{n_x}) \end{pmatrix} \quad \underline{F_y} = \begin{pmatrix} \sigma_{y_1}(y_0) \\ \sigma_{y_1}(y_1) \\ . \\ . \\ . \\ \sigma_{y_1}(y_{n_y}) \\ \sigma_{y_2}(y_0) \\ . \\ . \\ . \\ \sigma_{y_{N_x}}(y_{n_y}) \end{pmatrix}$$

The following steps show the formulation of condition (8) of the tau problem (6) to (8), let

$$\Pi_{x_q n_x n_y}(x, y) = \sum_{r_x=0}^{w_x} \sum_{r_y=0}^{w_y} (\underline{\underline{y_{n_y}}}(\eta_{n_y}^{r_y})' \otimes \xi_{r_x r_y}(x, y) \underline{\underline{x_{n_x}}}(\eta_{n_x}^{r_x})') \quad (16)$$

Substituting (16) into equation (8) gives

$$\Pi_{x_q n_x n_y}(x, y_q) vec(A_{n_x n_y}) = \sigma_{x_q}(x), \quad q = 1(1)N_y \tag{17}$$

Let x_i, *for* $i = 0(1)n_x$ be the $n_x + 1$ zeros of the Polynomial $V_{x,n_x+1}^{[a_x,b_x]}(x)$. Collocating the $n_x + 1$ zeros into equation (17), we have

$$\Gamma_x vec(A_{n_x n_y}) = \underline{F} \tag{18}$$

$$\Gamma_x = \begin{pmatrix} \Pi_{x_1 n_x n_y}(x_0, y_1) \\ \Pi_{x_1 n_x n_y}(x_1, y_1) \\ . \\ . \\ . \\ \Pi_{x_1 n_x n_y}(x_{n_x}, y_1) \\ \Pi_{x_2 n_x n_y}(x_0, y_2) \\ . \\ . \\ . \\ \Pi_{x_{N_x} n_x n_y}(x_{n_x}, y_{N_x}) \end{pmatrix} \quad \underline{F_x} = \begin{pmatrix} \sigma_{x_1}(x_0) \\ \sigma_{x_1}(x_1) \\ . \\ . \\ . \\ \sigma_{x_1}(x_{n_x}) \\ \sigma_{x_2}(x_0) \\ . \\ . \\ . \\ \sigma_{x_{N_y}}(x_{n_x}) \end{pmatrix}$$

By merging equations (12),(15) and (18), we have aresulting system of linear algebraic equations

$$\begin{pmatrix} \Gamma \\ \Gamma_y \\ \Gamma_x \end{pmatrix} vec(A_{n_x n_y}) = \begin{pmatrix} \underline{F} \\ \underline{F_y} \\ \underline{F_x} \end{pmatrix} \tag{19}$$

Where $(\Gamma, \Gamma_y, \Gamma_x)'$ *is a* $((n_x+1)(n_y+1) + N_x N_y) \times (n_x+1)(n_y+1)$ matrix and $(\underline{F}, \underline{F_y}, \underline{F_x})'$ is a column vector with $(n_x+1)(n_y+1) + N_x N_y)$ elements. Since there are $N_x N_y$ redundant linear dependent equations inside system (19), the rank of the system (19) is now $(n_x+1)(n_y+1)$ Therefore all of the unknown coefficients $a_{i_x i_y}, i_x = 0(1)n_x,$ *and* $i_y = 0(1)n_y$, can be obtained by solving the system of linear algebraic equation (19) through the usual method without finding out free parameters τ_x *and* τ_y since they have zero as there coefficients. The tau approximant $u_{n_x n_y}(x, y)$ for the solution of problem (3) to (5) can then be obtained.

FORMULATION Of THE NONLINEAR PART USING ADOMIAN'S POLYNOMIAL

The Adomian Polynomial originated from the Adomian decomposition method. The role it plays in the solution of nonlinear differential equations is to convert the nonlinear terms of the differential equations into a set of polynomials and it can be used in approximating the solution of nonlinear differential equations with highly nonlinear terms such as trig and exponential nonlinearity. The following is the process of linearization by Adomian polynomials.

Consider a two dimensional nonlinear PDE

$$Lu(x, y) + F(u(x, y)) = f(x, y), \quad (x, y) \in [a_x, b_x] \times [a_y, b_y] \tag{20}$$

Subject to the supplementary conditions

$$D_{y_p}\Big|_{x=x_p} u(x, y) = \sigma_{y_p}(y), \quad p = 1(1)N_x \tag{21}$$

$$D_{x_q}\Big|_{y=y_q} u(x, y) = \sigma_{x_q}(y), \quad q = 1(1)N_y \tag{22}$$

Where $F(u(x, y))$ is the nonlinear term of the above given problem. Let

$$u(x,y)=\sum_{i=0}^{\infty}u_{(i)}(x,y), \tag{23}$$

where $u_{(i)}(x,y), i=0(1)\infty$ are the decomposed solution of the problem (20)-(22) and let $u_{n_x n_y (i)}(x,y)$ be the numerical approximate solution to $u_{(i)}(x,y)$ with degree n_x and n_y $in\, x$ and y respectively for $i=0(1)\infty$. The nonlinear term $F(u(x,y))$ in equation (20) can be written in terms of the Adomian's polynomials following the method of ODEs.

$$F(u(x,y))=\sum_{i=0}^{\infty}A_i=\sum_{i=0}^{\infty}\frac{1}{i!}\frac{d^i}{d\lambda^i}F(\sum_{j=0}^{i}\lambda^j u_{(j)}(x,y)\Big|_{\lambda=0} \tag{24}$$

where $A_i=\frac{1}{i!}\frac{d^i}{d\lambda^i}F(\sum_{j=0}^{i}\lambda^j u_{(j)}(x,y)\Big|_{\lambda=0}$, $i=0(1)\infty$, are the Adonian's polynomials. Problem (20)-(22) can now be decomposed into infinitely many sub-problems by the principle of superposition and the problem becomes

$$\begin{cases} Lu_{(0)}(x,y)=f(x) \\ D_{y_p}\Big|_{x=x_p}u_{(0)}(x,y)=\sigma_{y_p}(y) \\ D_{x_q}\Big|_{y=y_q}u_{(0)}(x,y)=\sigma_{x_q}(x) \end{cases} \quad and \begin{cases} Lu_{(i+1)}(x,y)=-A_i \\ D_{y_p}\Big|_{x=x_p}u_{(i+1)}(x,y)=0 \\ D_{x_q}\Big|_{y=y_q}u_{(i+1)}(x,y)=0 \end{cases} \tag{25}$$

Where $p=1(1)N_x$, $q=1(1)N_y$ $and\, i=0(1)\infty$ .On completion of the linearization process, we can now handle the linearized equation as the ones treated above for linear PDEs..

III. NUMERICAL EXPERIMENT

3.1 Numerical Examples

Problem 1

Consider a second order 2-dimentional linear PDE

$$\frac{\partial^2}{\partial x\partial y}u(x,y)=4xy+e^x, \quad (x,y)\in[0,1]\times[0,1] \tag{20a}$$

With supplementary conditions

$$\frac{\partial}{\partial y}u(0,y)=y, \qquad y\in[0,1] \tag{20b}$$

$$u(x,0)=2, \qquad x\in[0,1] \tag{20c}$$

The exact solution of this problem is

$$u(x,y)=x^2y^2+ye^x+\frac{y^2}{2}-y+2$$

The following steps are the main procedure for the tau-Collocation method

1.Setup the tau-method:

If we take tau degree $n_x=n_y=2$ and use the Catalan basis in the perturbation term $H_{n_x n_y}(x,y)$, the Tau problem (20a) to (20c) becomes

$$\frac{\partial^2}{\partial x\partial y}u(x,y)=4xy+e^x+g_{22}(x,y;\tau_x,\tau_y)C_2^*(x)C_2^*(y), \tag{21a}$$

Where $(x,y)\in[0,1]\times[0,1]$ with supplementary conditions

$$\frac{\partial}{\partial y}u_{22}(0,y)=y, \quad y\in[0,1] \tag{21b}$$

$u_{22}(x,0) = 2, \quad x \in [0,1]$ (22c)

1. Formulate the matrices Γ *and* $\underline{F}$:

Since

$$\left((1\ y^1\ y^2)\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}' \otimes (1\ x^1\ x^2)\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}' \right) vec(A_{22})$$

$= (0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 2x\ \ 0\ \ 2y\ \ 4xy)\, vec(A_{22})$ (23)

By collocating the zeros of $C_2^*(x)$ *and* $C_2^*(y)$ *i.e* $(\frac{1}{3},\frac{1}{3}),(\frac{1}{3},\frac{2}{3}),$ $(\frac{2}{3},\frac{1}{3})$ *and* $(\frac{2}{3},\frac{2}{3}),$ into equation (23) and the right-hand side of equation (20a), we have

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & 0 & \frac{2}{3} & \frac{4}{9} \\ 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & 0 & \frac{4}{3} & \frac{8}{9} \\ 0 & 0 & 0 & 0 & 1 & \frac{4}{3} & 0 & \frac{2}{3} & \frac{8}{9} \\ 0 & 0 & 0 & 0 & 1 & \frac{4}{3} & 0 & \frac{4}{3} & \frac{16}{9} \end{pmatrix} \text{ and } \underline{F} = \begin{pmatrix} 1.84001035 \\ 2.28445479 \\ 2.83668785 \\ 3.72551831 \end{pmatrix}$$

2. Formulate the matrices Γ_y *and* $\underline{F_y}$

$$\Pi_{y_1 22}(0,y) = \begin{pmatrix} (1\ y\ y^2)\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}' \otimes \\ (1\ 0\ 0)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}' \end{pmatrix} vec(A_{22})$$

$$= (0\ 0\ 0\ 1\ 0\ 0\ 2y\ 0\ 0)\, vec(A_{22}) \quad (24)$$

By collocating the zeros of $C_3^*(y)$ *i.e* $\frac{1}{4},\frac{2}{4},$ *and* $\frac{3}{4}$ into equation (24) and the right-hand side of equation (21b)

$$\Gamma_y = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1.5 & 0 & 0 \end{pmatrix} \text{ and } \underline{F_y} = \begin{pmatrix} 0.25 \\ 0.50 \\ 0.75 \end{pmatrix}$$

4. Formulate the matrices Γ_x *and* $\underline{F_x}$. Since

$$\Pi_{x_1 22}(x,0) = \begin{pmatrix} (1\ 0\ 0)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}' \otimes \\ (1\ x\ x^2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}' \end{pmatrix} vec(A_{22}) = (1\ x\ x^2\ 0\ 0\ 0\ 0\ 0\ 0)vec(A_{22}) \tag{25}$$

By collocating the zeros of $C_3^*(x)$ *i.e* $\frac{1}{4}, \frac{2}{4}$ *and* $\frac{3}{4}$ into equation (25) and the right-hand side of equation (21c), we have

$$\Gamma_x = \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{3}{4} & \frac{9}{16} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \underline{F_x} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

5. Formulate the system of linear equation and solve it to obtain the Tau approximant.

By combining matrices Γ, Γ_y *and* Γ_x and column vectors $\underline{F}, \underline{F_y}$ *and* $\underline{F_x}$ we obtain

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & 0 & \frac{2}{3} & \frac{4}{9} & 1.84001035 \\ 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & 0 & \frac{4}{3} & \frac{8}{9} & 2.28445479 \\ 0 & 0 & 0 & 0 & 1 & \frac{4}{3} & 0 & \frac{2}{3} & \frac{8}{9} & 2.836687 \\ 0 & 0 & 0 & 0 & 1 & \frac{4}{3} & 0 & \frac{4}{3} & \frac{16}{9} & 3.72551831 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0.25 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{3}{2} & 0 & 0 & 0.75 \\ 1 & \frac{1}{4} & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & \frac{3}{4} & \frac{9}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

By solving the above matrix using the Gauss Elimination Method we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0.84327613 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0.8284346700 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.9998704576 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (26)$$

Note that the last row of matrix is a zero row which is the redundant linear dependent equation generated from the nearby conditions (21b) and (21c) during the collocation process. From matrix (26) we can obtain the Tau approximant

$$u_{22}(x, y) = 2 + 0.84328\ xy + 0.82843\ x^2 y + 0.5y^2 + 0.07009\ y^2 x + 0.99987\ x^2 y^2 \quad (27)$$

The dimension of the matrix is determined by the degree of Catalan polynomial used. These obtained values of $a_{i_x i_y}$ are substituted in (2) to obtain the required approximate solution.

We then compare the errors of the new method with that of Sam and Liu (2004). , The error is defined as

$$Error = \left| u_{exact}(x, y) - u_{i_x i_y} \right| \quad (28)$$

If we take tau degree $n_x = n_y = 3$ and use Catalan polynomial basis in the perturbation term $H_{n_x n_y}$, we obtain the Tau approximant

$$u_{33}(x, y) = 2 + 1.022912\ yx + 0.418643\ yx^2 + 0.276221\ yx^3 + 0.5y^2 + y^2 x^2 \quad (29)$$

Problem2:

Consider a nonlinear PDE

$$\frac{\partial}{\partial t} u(x,t) - \frac{\partial^2}{\partial t^2} u(x,t) - (u(x,t))^2 = f(x,t), \quad (x,t) \in [0,1] \times [0,1] \quad (30)$$

Where

$$f(x,t) = e^t \sin \pi x (1 + \pi^2 - e^t \sin \pi x) \quad (31)$$

With initial conditions

$$u(0,t) = 0 \quad (32)$$

$$u(1,t) = 0, \quad t \in [0,1] \quad (33)$$

The exact solution of the problem is

$$u(x,t) = e^t \sin \pi x$$

The Adomian's polynomial technique of linearization can also be applied to the problem above and we obtain the following as its polynomials for values of $k = 1, 2, 3, \ldots,$

$$A_0 = \left(\frac{d}{dx} u_0(x) \right)^2$$

$$A_1 = 2\left(\frac{d}{dx}u_0(x)\right)\frac{d}{dx}u_1(x)$$

$$A_2 = \left(\frac{d}{dx}u_1(x)\right)^2 + 2\left(\frac{d}{dx}u_0(x)\right)\frac{d}{dx}u_2(x)$$

$$A_3 = 2\left(\frac{d}{dx}u_1(x)\right)\frac{d}{dx}u_2(x) + 2\left(\frac{d}{dx}u_0(x)\right)\frac{d}{dx}u_3(x)$$

The Tau-collocation method can now be applied directly to this problem without any pre-approximation to the right hand sides of both problem and its conditions. Going through the same process of formulation as in examples (1) above we arrive at a result displayed on table 3.

Table 1: Comparing the absolute Errors in the new method to Errors in Sam (2004) for example 1

		At Tau degree $n_x = n_y = 2$		At Tau degree $n_x = n_y = 3$	
i_x	i_y	New Method	Sam(2004)	New Method	Sam (2004)
0.0	0.0	0.000000E-00	0.000000E-00	0.000000E-00	0.000000E-00
0.1	0.1	5.685875E-03	5.685343E-03	2.460000E-04	2.456325E-04
0.2	0.2	1.936212E-03	1.925437E-03	2.694000E-03	2.765321E-03
0.3	0.3	3.630506E-03	2.225205E-03	9.505000E-03	9.670361E-03
0.4	0.4	5.230889E-02	2.927365E-02	2.223470E-02	2.132537E-02
0.5	0.5	6.374501E-02	8.644801E-03	4.246700E-02	5.231860E-03
0.6	0.6	6.764645E-02	1.655559E-02	7.076700E-02	5.209126E-02
0.7	0.7	6.178942E-02	2.554448E-02	1.078840E-02	1.024871E-02
0.8	0.8	4.479063E-02	3.336561E-02	1.542930E-02	1.664588E-01
0.9	0.9	1.621582E-02	3.798617E-02	2.004120E-01	2.022398E-01
1.0	1.0	2.329817E-02	3.455992E-01	2.767270E-01	2.523652E-01

Table 2: Comparing the absolute Errors in the new method to Errors in Sam (2004) for example 1

		At Tau degree $n_x = n_y = 4$		At Tau degree $n_x = n_y = 5$	
i_x	i_y	New Method	Sam (2004)	New Method	Sam(2004)
0.0	0.0	0.000000E-00	0.000000E-00	0.000000E-00	0.000000E-00
0.1	0.1	1.675431E-05	1.675722E-05	1.321001E-06	1.323043E-06
0.2	0.2	3.542163E-05	3.546254E-05	2.061424E-06	2.062190E-06
0.3	0.3	2.435630E-04	2.413268E-04	7.182013E-05	7.016920E-05
0.4	0.4	7.841066E-03	6.893126E-03	3.325445E-04	3.248903E-04
0.5	0.5	6.432502E-03	4.217657E-04	8.320876E-04	7.960231E-05
0.6	0.6	8.123868E-03	8.143256E-03	5.321084E-04	5.256174E-04
0.7	0.7	7.216987E-03	7.113249E-03	2.653452E-04	2.545438E-04
0.8	0.8	1.528701E-02	1.600821E-02	3.132505E-03	3.234320E-03
0.9	0.9	1.467290E-02	1.463219E-02	2.333333E-03	2.321546E-03
1.0	1.0	1.031453E-02	1.042945E-02	9.216589E-03	9.235672E-03

Table 3: Comparing the absolute Errors in the new method to Errors in Sam (2004) in example 2

		At Tau degree $n_x = n_t = 3$		At Tau degree $n_x = n_t = 4$	
i_x	i_t	New Method	Sam(2004)	New Method	Sam (2004)
0.0	0.0	0.000000E-00	0.000000E-00	0.000000E-00	0.000000E-00
0.1	0.1	3.215875E-07	6.385343E-07	4.220000E-09	5.456325E-09
0.2	0.2	1.036212E-07	3.925437E-06	1.654000E-09	0.765321E-08
0.3	0.3	4.530506E-06	5.225205E-06	6.525000E-08	9.300361E-08
0.4	0.4	1.200889E-06	0.927365E-05	1.513470E-08	2.150537E-08
0.5	0.5	3.574501E-05	1.644801E-05	5.346700E-07	7.111860E-07
0.6	0.6	5.394645E-05	2.455559E-04	6.286700E-07	8.319126E-07
0.7	0.7	0.548942E-04	4.554448E-04	3.058840E-07	1.754871E-06
0.8	0.8	3.479063E-04	5.336561E-04	0.702930E-06	3.534588E-06
0.9	0.9	3.551582E-03	6.700617E-03	1.114120E-05	4.522398E-05
1.0	1.0	3.029817E-02	5.395992E-02	0.217270E-04	1.433652E-04

IV. DISCUSSION OF RESULTS

The approximate solutions obtained from these experiments shows the efficiency of the method. It is observed from the tables that the result obtained from the Catalan tau collocation method converges faster as the degree of tau increases with a decrease in step number.Generally, the performance of our method as seen on the tables above, are superior to those from tau collocation method using Chebyshev as a polynomial basis function by Sam and Liu (2004) for the same degree of tau and step length.

Tables 1and 2 are the solution for 2-dimentional linear PDEs at varied degrees of tau. From the tables the new collocation approach is seen competing favourably and even better at some instances to that of Sam (2004).

Table 3 and 4 are the result for solving 3-dimentional linear PDE problem. There is also an increasing level of accuracy with a decrease in step length at varied degree of tau. The performance of Catalan Polynomial as a basis function here is also quite commendable if one should consider Chebyshev polynomial being a standard and most widely used and acceptable basis function.

V. CONCLUSION

From the presentations above, we have been able to develop a method using Catalan polynomial basis to solve partial differential equations directly, this is an attempt to introduce the use of Catalan polynomials into the numerical solution of partial differential equations directly and the result competes well with Chebyshev polynomial basis function which is a widely used and acceptable polynomial basis function.

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